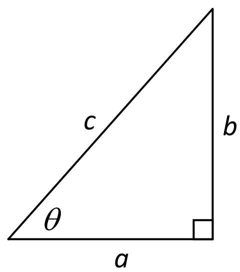
Lesson 5: The Six Trigonometric Functions

The word “trigonometry” means “measuring triangles.” We introduce the trigonometric functions in the context of *right* triangles.

In the right triangle below, we can form six different ratios of two sides.



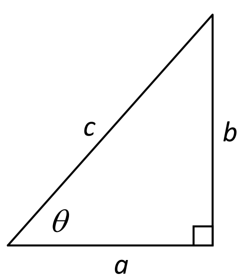
Six possible ratios:

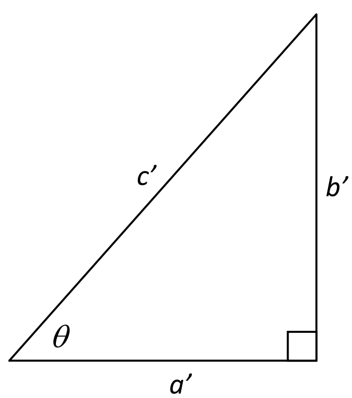
$\frac{b}{c}$ $\quad \frac{c}{b}$

$\frac{a}{c}$ $\quad \frac{c}{a}$

$\frac{b}{a}$ $\quad \frac{a}{b}$

Note that these ratios only depend on the angle θ (which determines the “shape” of the triangle). The ratios do not depend on the size of the triangle. This is because if two triangles have the same angles but are different sizes then they are similar and the corresponding ratios are equal:



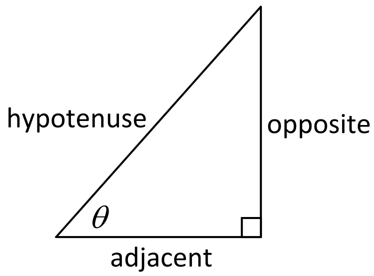


$\frac{b}{c} = \frac{b'}{c'}$ $\quad \frac{c}{b} = \frac{c'}{b'}$

$\frac{a}{c} = \frac{a'}{c'}$ $\quad \frac{c}{a} = \frac{c'}{a'}$

$\frac{b}{a} = \frac{b'}{a'}$ $\quad \frac{a}{b} = \frac{a'}{b'}$

The six ratios mentioned above that depend on the angle θ are the six trigonometric functions of θ named as follows:



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

We've used more suggestive names for the sides of the triangle than *a*, *b*, and *c*. This is important because the triangle isn't always oriented the same way and the angle in which we're interested isn't always in the lower left corner.

Note that we are using function notation. Writing $\cos(\theta)$ is analogous to writing $f(x)$. The name of the function is cos and the input is θ. The output is the given ratio in each case. The names of the six trigonometric functions above are abbreviations for the following words:

cos = cosine
sin = sine
tan = tangent

sec = secant
csc = cosecant
cot = cotangent

For us, the choice of names for these six ratios is arbitrary. (Although later, we will see why some of them have "co" in front of another name.) So, we must purely memorize them. A mnemonic device helps. A popular one is:

"SOH CAH TOA"

The letters in SOH indicate: **<u>S</u>**in = **<u>O</u>**pposite / **<u>H</u>**ypotenuse
The letters in CAH indicate: **<u>C</u>**os = **<u>A</u>**djacent / **<u>H</u>**ypotenuse
The letters in TOA indicate: **<u>T</u>**an = **<u>O</u>**pposite / **<u>A</u>**djacent

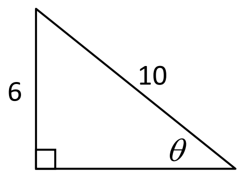
This gives you a way to remember three of the ratios. The other three ratios are reciprocals of these, but you will have to remember which function is the reciprocal of which:

$\sec(\theta)$ is the reciprocal of $\cos(\theta)$

$\csc(\theta)$ is the reciprocal of $\sin(\theta)$

$\cot(\theta)$ is the reciprocal of $\tan(\theta)$

Example Find $\tan(\theta)$ in the following triangle.

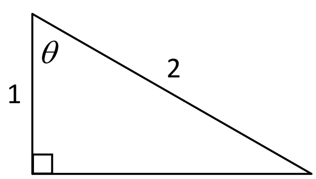


Solution First we find the missing side. Multiplying the Pythagorean triple 3-4-5 by 2 gives 6-8-10. Therefore, the missing side is 8. (Alternatively, we may use the Pythagorean Theorem.) Next, we recall the definition of $\tan(\theta)$ (think “SOH CAH TOA”). It is:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

In this triangle, the side opposite θ is 6. The adjacent side is 8. So, $\tan(\theta) = \frac{6}{8} = \frac{3}{4}$.

Example Find $\csc(\theta)$ in the following triangle.



Solution First, we find the missing side. Call it a. Then

$$a^2 + 1^2 = 2^2$$

$$a^2 = 3$$

$$a = \sqrt{3}$$

Then, since $\csc(\theta) = \text{hypotenuse / opposite}$, we have

$$\csc(\theta) = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$