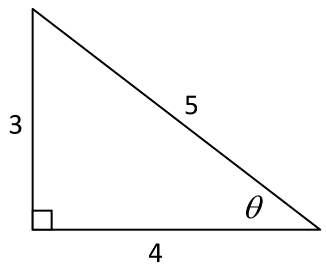
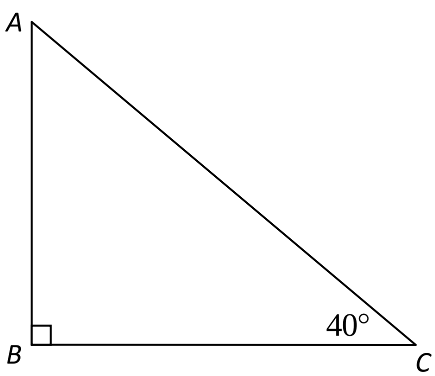
## Lesson 6: Special Triangles

We want to stress that the input to a trigonometric function is an acute angle $\theta$. To compute a trigonometric function of $\theta$ we draw a right triangle with that angle $\theta$. For example, suppose $\theta$ is the acute angle depicted in the triangle below.




Then $\tan(\theta) = 3/4 = 0.75$. However, note that we have computed $\tan(\theta)$ without even knowing what the input $\theta$ is. In the triangle above, it turns out that $\theta \approx 36.87°$. So, we have discovered that $\tan(36.87°) \approx 0.75$.

But what do we do if we have to compute $\tan(40°)$? Well, we could get out a protractor and measure off a 40 degree angle and complete a right triangle like this:




Note that it doesn't matter what size you make the triangle since corresponding ratios are equal in similar triangles. Now we can measure the sides *AB* and *BC* and divide them:

$$\tan(40°) = AB / BC$$

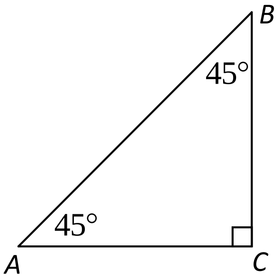
Of course, no one actually uses a ruler and protractor like this to compute $\tan(40°)$. There are advanced algorithms for finding the answer, and luckily our calculators know these algorithms. Verify on your calculator that

$$\tan(40°) \approx 0.8391$$

We do not always have to turn to our calculators or advanced algorithms to evaluate trigonometric functions. There are some common angles that we can deal with by hand. They are $30°, 45°,$ and $60°$.

**The 45-45-90 Triangle**

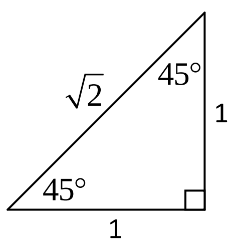
If we want to compute a trigonometric function of $45°$, we would draw a right triangle with a $45°$ angle. Note that since the three angles sum to $180°$, the other acute angle is $45°$ too.




Note that since angles *A* and *B* are equal, the sides opposite these angles are equal: *AC* = *BC*. Remember that for the purpose of computing the trigonometric functions, it doesn't matter what size this triangle is. So, choose *AC* = *BC* = 1. Let *c* denote the length of the hypotenuse and use the Pythagorean theorem to find it:

$$c^2 = 1^2 + 1^2 = 2, \quad c = \sqrt{2}$$

Therefore, there is a 45-45-90 triangle with the following side lengths:




Remember this triangle—we will use it a lot. Actually, there's not much to remember. Just choose 1 for the equal legs and compute the hypotenuse using the Pythagorean theorem.

**Example** Find the six trigonometric functions of $45°$.

**Solution** Looking at the 45-45-90 triangle above, we have (do not memorize these answers):

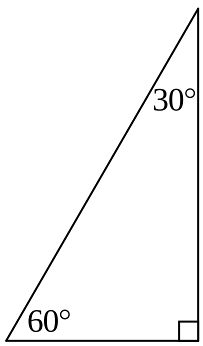
$$\sin(45°) = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \csc(45°) = \frac{\sqrt{2}}{1} = \sqrt{2}$$

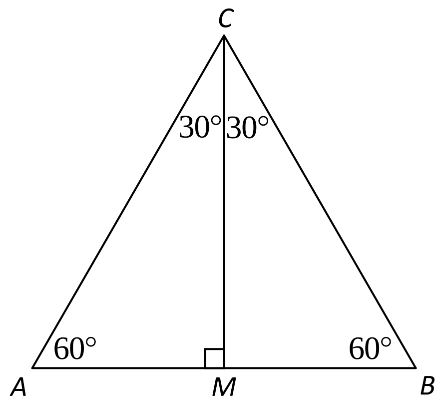
$$\cos(45°) = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \sec(45°) = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan(45°) = \frac{1}{1} = 1 \qquad \cot(45°) = \frac{1}{1} = 1$$

## The 30-60-90 Triangle

If we want to compute a trigonometric function of $30^\circ$ or $60^\circ$, we would draw a right triangle with angles as marked in the triangle below to the left.

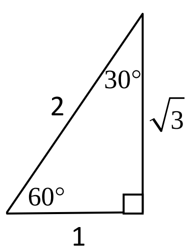



To find out how the sides relate to each other, we reflect the triangle over its longer (vertical) leg to produce the equilateral triangle above to the right. It is equilateral because all of its angles are $60^\circ$. Again, it doesn't matter what size the triangle is. For convenience, let each side of the equilateral triangle be 2. Note that $M$ is the midpoint of $AB$. So, considering triangle $AMC$, we have $AC = 2$ and $AM = 1$. Let $MC = x$ and apply the Pythagorean theorem:

$$x^2 + 1^2 = 2^2$$
$$x^2 = 3$$
$$x = \sqrt{3}$$

Therefore, we have the following 30-60-90 triangle. Remember it.




In particular, remember that the side opposite the $30^\circ$ angle is half the hypotenuse. This is important because when we work with this triangle it isn't always oriented like the one to the left.

**Example** Find the six trigonometric functions of $30^\circ$ and of $60^\circ$.
**Solution** Referring to the 30-60-90 triangle above, we have (do not memorize these answers):

$$\sin(60^\circ) = \frac{\sqrt{3}}{2} \qquad \csc(60^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sin(30^\circ) = \frac{1}{2} \qquad \csc(30^\circ) = \frac{2}{1} = 2$$

$$\cos(60^\circ) = \frac{1}{2} \qquad \sec(60^\circ) = \frac{2}{1} = 2 \qquad \cos(30^\circ) = \frac{\sqrt{3}}{2} \qquad \sec(30^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan(60^\circ) = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \cot(60^\circ) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan(30^\circ) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot(30^\circ) = \frac{\sqrt{3}}{1} = \sqrt{3}$$